

Linear-algebraic λ -calculus

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Abstract

With a view towards models of quantum computation and/or the interpretation of linear logic, we define a functional language where all functions are linear operators by construction. A small step operational semantic (and hence an interpreter/simulator) is provided for this language in the form of a term rewrite system. The *linear-algebraic λ -calculus* hereby constructed is linear in a different (yet related) sense to that, say, of the linear λ -calculus. These various notions of linearity are discussed in the context of quantum programming languages.

1 Introduction

Quantum computation lacks a convenient model of computation. To this day its algorithms are expressed in terms of quantum circuits, but their descriptions always seem astonishingly remote from the task they do accomplish [12]. Moreover universality is only provided via the notion of uniform family of circuits [28]. Quantum Turing machines solve this latter point, yet they are even less suitable as a programming language [6]. Another approach is to enclose quantum circuits within a classical imperative-style control structure [20] — but we wish to avoid this duality, in an attempt to bring programs closer to their specifications. Functional-style control structure, on the other hand, seem to merge with quantum evolution descriptions in a unifying manner. With a view towards models of quantum computation, we describe a functional language for expressing linear operators, and linear operators only.

We are careful, however, not to bury our presentation of this language of linear operators within too many quantum computation-specific considerations. The aim is to reach an audience of logicians also, as we suspect a strong connection with issues of computational interpretations of linear logic.

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We provide a semantic for the language in the form of a term rewrite system [11]. These consist in a finite set of rules $l \longrightarrow r$, each interpreted as follows: “Any term t containing a subterm σl in position p (i.e. $t = t[\sigma l]_p$) should be rewritten into a term t' containing σr in position p , with all the rest unchanged (i.e. $t' = t[\sigma r]_p$)”. Here σ denotes a variable substitution. The minimalist interpretation of the rules makes term rewrite systems (TRS) extremely suitable for describing the behavior of a computer languages unambiguously — so long as the order in which the reductions occur does not matter to the end result (a property named confluence). Moreover, because $l \longrightarrow r$ may be seen as an oriented version of equation $l = r$, the TRS provides both an operational semantic (an interpreter/simulator for the language) and an axiomatic semantic (an equational theory in which to prove properties about the language).

We begin with a simple language for vectors containing constants for base vectors, addition and product by a scalar. On terms of this language we define a rewrite system reducing any term expressing a vector to a linear combination of base vectors [4]. We have also proposed in [4] an extension to a language containing a tensorial product operation (Section 2).

Such a language must rely on a language and rewrite system for scalars. This raises the problem of the conditional rewriting required for division, which we can circumvent, basing quantum computation upon the ring of diadic floats together with $\frac{1}{\sqrt{2}}$ and imaginary number i (Section 3). More generally, it should be said that a language of linear operators does not need division.

Modern days functional languages such as Caml, Haskell etc. are based upon two basic evaluation mechanisms: *matching*, which provides conditional branching by inspection of values; and some avatar of the λ -calculus. The first mechanism is obtained as we extend the term rewrite system to handle linear maps — themselves denoted as superpositions of bipartite states, e.g.

$$(\mathbf{true} \triangleright \mathbf{false} + \mathbf{false} \triangleright \mathbf{true}) * \mathbf{false} \longrightarrow^* \mathbf{true}.$$

Applications are therefore analogous to contractions in tensorial calculus: this approach offers an elegant paradigm to represent quantum operations as quantum states (Section 4).

The second mechanism is obtained through an implementation of λ -terms via de Bruijn indices, a scheme whereby variables are encoded as integers referring to their binders, e.g.

$$\lambda x.(\lambda y.(x \otimes y)) \text{ is encoded as } L(L(\mathbf{var}(1) \otimes \mathbf{var}(0))).$$

The question of the interpretation of terms such as $\lambda x.(x \otimes x)$ is lengthily addressed as we draw a distinction between *cloning* and *copying*. The semantic of our calculus forbids only the former, non-linear operation, by enforcing a higher priority of the addition’s distributivity over substitution (Section 5). This is followed by a short example taken from our implementation (Section 6).

Erasure on the other hand remains allowed in our calculus, because we do not restrict ourselves to unitary operations. Whilst we discuss possible well-formedness conditions to implement this restriction (a crucial one for quantum computation), the claim here is to have provided a “*linear*” λ -calculus, in the sense of *linear algebra*. We discuss the various notions of “linearity” used in quantum programming languages, such as the one by Van Tonder [23] (Section 7).

2 Vectorial spaces

We seek to represent quantum programs, their input vectors, their output vectors and their applications as terms of a first-order language. Moreover we seek to provide rules such that the term formed by the application of a quantum program onto its input vector should reduce to its output vector. Several terms may be used to express one output vector, as a consequence we must ensure that these all reduce to one unique, *normal form*, upon which there is nothing more to compute. The most natural normal form to aim for is that of a linear combination of the base vectors, i.e. the computation finishes once we have the coordinates of the output vector.

We start with the language of vectorial spaces, i.e. a two-sorted language \mathcal{L} having sort K for scalars and sort E for vectors — together with: two constants 0 and 1 of sort K ; a constant $\mathbf{0}$ of sort E ; two binary symbols $+$ and \times of rank $\langle K, K, K \rangle$; a binary symbol $+$ (also) of rank $\langle E, E, E \rangle$; and a binary symbol $.$ of rank $\langle K, E, E \rangle$. In [4] we described a term rewrite system reducing any term expressing a vector into a linear combination of base vectors. The term rewrite system develops

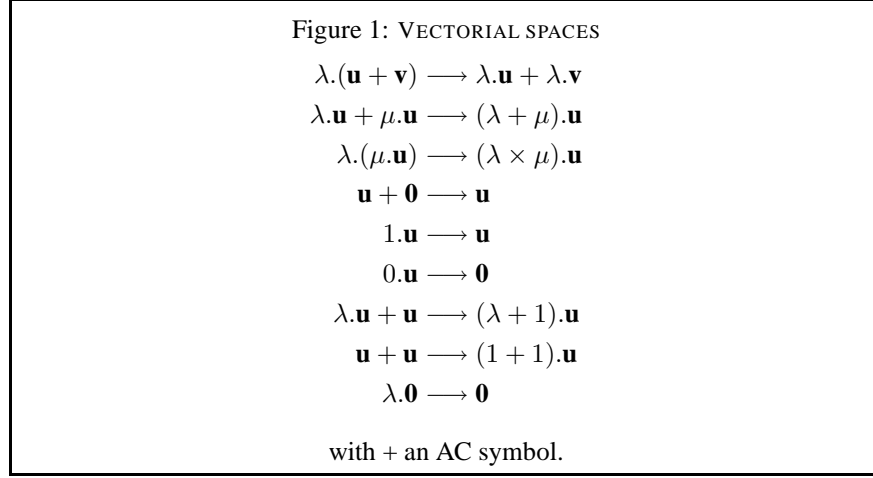
$$4.(\mathbf{false} + \mathbf{true}) \longrightarrow 4.\mathbf{false} + 4.\mathbf{true}$$

but factorizes

$$4.\mathbf{false} + 6.\mathbf{false} \longrightarrow (4 + 6).\mathbf{false}.$$

according to the rules in figure 1. Such a TRS arises as we orient six of the eight equations axiomatizing vectorial spaces. Only those two axioms corresponding to associativity and commutativity of vector addition are left aside, because we use

rewriting modulo AC(+). Moreover we need to add three more rules for confluence.



But these rewrite rules do not take into account computation on scalars. The latter must be added by mixing in another rewrite system S , rewriting scalar to a normal form.

Definition 1 (*Scalar rewrite system*) A scalar rewrite system is a rewrite system on a language containing at least the symbols $+$, \times , 0 and 1 , such that:

- S is terminating and ground confluent,
- for all closed terms λ , μ and ν , the pair of terms
 - $0 + \lambda$ and λ ,
 - $0 \times \lambda$ and 0 ,
 - $1 \times \lambda$ and λ ,
 - $\lambda \times (\mu + \nu)$ and $(\lambda \times \mu) + (\lambda \times \nu)$,
 - $(\lambda + \mu) + \nu$ and $\lambda + (\mu + \nu)$,
 - $\lambda + \mu$ and $\mu + \lambda$,
 - $(\lambda \times \mu) \times \nu$ and $\lambda \times (\mu \times \nu)$,
 - $\lambda \times \mu$ and $\mu \times \lambda$

have the same normal forms,

- 0 and 1 are normal terms.

The following propositions can be found in [4].

Proposition 1 *For any scalar rewrite system S , the rewrite system $R \cup S$ is terminating and ground confluent.*

Proposition 2 *If t is a normal close term whose constants are amongst $\mathbf{x}_1, \dots, \mathbf{x}_n$. The term t is $\mathbf{0}$ or it has the form $\lambda_1 \mathbf{x}_{i_1} + \dots \lambda_k \mathbf{x}_{i_k} + \mathbf{x}_{i_{k+1}} + \mathbf{x}_{i_{k+l}}$ where the indices i_1, \dots, i_{k+l} are distinct and the λ_k 's are neither 0 nor 1.*

Note that the algorithm defined by R is relatively common in computing, for presenting any vector as a linear combination of base vectors. But it does in fact define vectorial spaces, as any mathematical structure validating the algorithm. In this sense we have provided a computational definition of vectorial spaces.

Furthermore note that the support for tensor products is easily added into the TRS, through the six rules given in figure 2. Proposition 1 remains true when R is extended with those six additional rules, whilst proposition 2 now yields normal forms for terms in $E \otimes E$ of the form $\mathbf{0}$ or

$$\lambda_1 \mathbf{x}_{i_1} \otimes \mathbf{y}_{j_1} + \dots + \lambda_k \mathbf{x}_{i_k} \otimes \mathbf{y}_{j_k} + \mathbf{x}_{i_{k+1}} \otimes \mathbf{y}_{j_{k+1}} + \dots + \mathbf{x}_{i_{k+l}} \otimes \mathbf{y}_{j_{k+l}},$$

where the pairs of indices $\langle i_1, j_1 \rangle, \dots, \langle i_{k+l}, j_{k+l} \rangle$ are distinct and the λ_k 's are neither 0 nor 1 [4].

Figure 2: VECTORIAL SPACES: TENSORS

$$(\mathbf{u} + \mathbf{v}) \otimes \mathbf{w} \longrightarrow \mathbf{u} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}$$

$$(\lambda.\mathbf{u}) \otimes \mathbf{v} \longrightarrow \lambda.(\mathbf{u} \otimes \mathbf{v})$$

$$\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) \longrightarrow \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{w}$$

$$\mathbf{u} \otimes (\lambda.\mathbf{v}) \longrightarrow \lambda.(\mathbf{u} \otimes \mathbf{v})$$

$$\mathbf{0} \otimes \mathbf{u} \longrightarrow \mathbf{0}$$

$$\mathbf{u} \otimes \mathbf{0} \longrightarrow \mathbf{0}$$

3 The field of quantum computing

Fields are not easily implemented as term rewrite systems, because of the conditional rewriting required for the division by zero. In the previous section such problems were avoided by simply assuming a TRS for scalars having a certain number of properties, but if the objective is to lay the ground for formal quantum programming languages, then we must provide such a TRS. The present section briefly outlines how this is achieved.

3.1 Background

We seek to model quantum computation as a formal rewrite system upon a finite set of symbols. Since the complex numbers are uncountable, we must therefore depart from using the whole of \mathbb{C} as the field \mathbb{K} of our vectorial space. Such considerations are commonplace in computation theory, and were successfully addressed with the provision of the first rigorous definition of a quantum Turing machine [6]. In short the quantum Turing machines are brought as an extension of probabilistic Turing machines

$$\begin{aligned} & \langle Q : \text{head states}, \Sigma : \text{alphabet}, \\ & \delta : \text{transition function}, q_o, q_f : \text{start, end state} \rangle \end{aligned}$$

whose transition functions are no longer valued over the efficiently computable positive reals (probabilities)

$$\delta : Q \times \Sigma \longrightarrow (Q \times \Sigma \times \{Left, Right\} \rightarrow \tilde{\mathbb{R}}^+)$$

but over the efficiently computable complex numbers (amplitudes)

$$\delta : Q \times \Sigma \longrightarrow (Q \times \Sigma \times \{Left, Right\} \rightarrow \tilde{\mathbb{C}}).$$

In both cases δ is constrained to be a *unit* function (probabilities/squared modulus summing to one), and for the quantum Turing machine δ is additionally required to induce a *unitary* global evolution. A well-known result of complexity theory is that probabilistic Turing machines remain as powerful when the transition function δ is further restricted to take values in the set $\{0, \frac{1}{2}, 1\}$. The result in [6] is analogous: quantum Turing machines remain as powerful when the transition function δ is further restricted to take values in the set $\{-1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\}$. Later it was shown in [2], and independently in [22] that no irrational number is necessary, i.e. δ may be restricted to take values in the set $\{-1, -\frac{8}{5}, -\frac{3}{5}, 0, \frac{3}{5}, \frac{8}{5}, 1\}$ without loss of power for the quantum Turing machine.

In the circuit model of quantum computation the emphasis was placed on the ability to *approximate* any unitary transform from a finite set of gates. This line of research (cf. [21][15] to cite a few) has so far culminated with [7], where the following set

$$\begin{aligned} CNOT &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ H &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \end{aligned} \quad (1)$$

was proven to be universal in the above strict sense. A weaker requirement for a set of gates is the ability to *simulate* any unitary transform, a notion which is also referred to as *encoded universality* — since a computation on n qubits may for instance be represented as a computation on $n + 1$ “real bits”, through a simple mapping. A recent paper shows that the gate

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix},$$

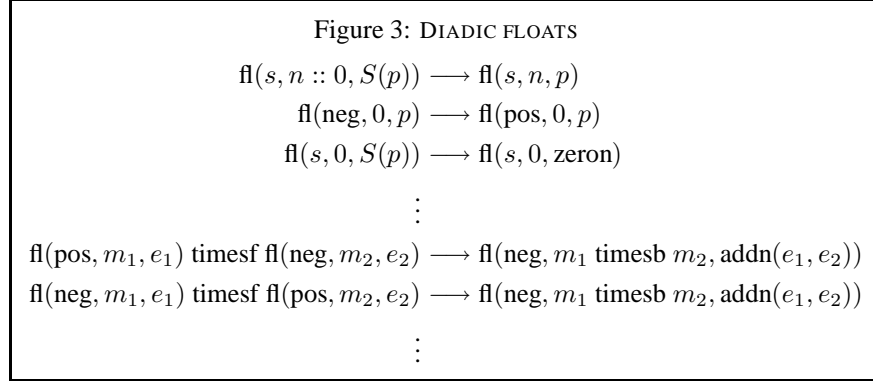
with either $a = b = \frac{1}{\sqrt{2}}$, or $a = \frac{3}{5}$ and $b = \frac{8}{5}$, has this property [19]. Do appreciate how the result falls into line with those regarding the quantum Turing machine.

Definition 2 We call computational scalars, and denote $\tilde{\mathbb{K}}$ the ring formed by the additive and multiplicative closure of the complex numbers $\{-1, 1, \frac{1}{\sqrt{2}}, i\}$.

Once we have shown that the computational scalars arithmetics can be performed by a TRS, it will be sufficient to express the basic gates (1) in our formalism to immediately obtain the more traditional notion of quantum computation universality. Hence our choice.

3.2 Rules

We begin by implementing natural numbers and unsigned binary numbers. That such TRS can be made ground confluent and terminating are now well-established results [9][27]. This places us in a position to build up diadic floats out of a sign,



an unsigned binary number and an exponent, e.g. $\text{fl}(\text{neg}, 1, S(\text{zeron}))$ is to stand for $-\frac{1}{2}$, as exemplified in figure 3.

Reached this point it suffices to notice that $\tilde{\mathbb{K}}$, i.e. diadic floats together with imaginary number i and real number $\frac{1}{\sqrt{2}}$, can be viewed as a four-dimensional module upon diadic floats. Indeed any such number could be represented as a linear combination of the form:

$$\alpha \cdot \mathbf{1} + \beta \cdot \frac{\mathbf{1}}{\sqrt{2}} + \gamma \cdot \mathbf{i} + \delta \cdot \frac{\mathbf{i}}{\sqrt{2}}.$$

As a consequence we can reuse the results of section 2 to implement computational scalars and their additions. Computational scalars multiplication then needs to be defined, we do so modulo AC in figure 4. Notice that we overload the symbol \times for multiplication of diadic floats and for multiplication of computational scalars.

We conjecture that this TRS is ground confluent and terminating, but have not yet a formal proof for this assertion.

Notice we have never defined a division operation. This is because only the ring properties of these numbers are required for expressing linear operations: we place ourselves upon a “module” rather than a full vectorial space.

4 Matching construct

We now turn to the definition of the matching constructs in our language. As we shall see, these constructs are nothing else than a reformulation of the rules for the

Figure 4: SCALAR MULTIPLICATION

$$\begin{aligned}
\mathbf{1} \times \mathbf{v} &\longrightarrow \mathbf{v} \\
\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} &\longrightarrow \text{fl}(\text{pos}, 1, S(\text{zeron})).1 \\
\frac{1}{\sqrt{2}} \times \mathbf{i} &\longrightarrow \frac{\mathbf{i}}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \times \frac{\mathbf{i}}{\sqrt{2}} &\longrightarrow \text{fl}(\text{pos}, 1, S(\text{zeron})).\mathbf{i} \\
\mathbf{i} \times \mathbf{i} &\longrightarrow \text{fl}(\text{neg}, 1, \text{zeron}).1 \\
\mathbf{i} \times \frac{\mathbf{i}}{\sqrt{2}} &\longrightarrow \text{fl}(\text{neg}, 1, \text{zeron}).\frac{1}{\sqrt{2}} \\
\frac{\mathbf{i}}{\sqrt{2}} \times \frac{\mathbf{i}}{\sqrt{2}} &\longrightarrow \text{fl}(\text{neg}, 1, S(\text{zeron})).1 \\
(\lambda.\mathbf{u}) \times \mathbf{v} &\longrightarrow \lambda.(\mathbf{u} \times \mathbf{v}) \\
(\mathbf{t} + \mathbf{u}) \times \mathbf{v} &\longrightarrow \mathbf{t} \times \mathbf{v} + \mathbf{u} \times \mathbf{v}
\end{aligned}$$

with \times an AC symbol.

tensor product.

4.1 Notations

Your typical functional language (Haskell, ML...) will always have “matching” constructs (for branching). For instance, here is a piece of Caml:

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let rec not b = match b with
| false -> true
| true  -> false ;;

```

We wish to provide such constructs in our linear-algebraic calculus. Strangely enough these matching constructs are very close to the tensorial product constructs.

Mathematicians and physicist in this field would write linear maps instead: $\text{NOT} = |\text{true}\rangle\langle\text{false}| + |\text{false}\rangle\langle\text{true}|$. However here the $\langle\text{false}|$ and $\langle\text{true}|$ may be viewed as patterns, waiting to be compared to the input vector through a scalar

product. Thus we choose to reconcile both worlds and write:

$$\text{NOT} = \text{false} \triangleright \text{true} + \text{true} \triangleright \text{false}.$$

An expression $(\mathbf{t} \triangleright \mathbf{u})$ applied to a vector \mathbf{v} will then reduce into $(\mathbf{t} \bullet \mathbf{v}).\mathbf{u}$, with \bullet the scalar product. In this sense $(\mathbf{t} \triangleright \mathbf{u}) * \mathbf{v}$ does return \mathbf{u} *in so far as* \mathbf{t} overlaps with \mathbf{v} . More formal justifications, and formal rewrite rules follow in the next two subsections. For now we give the reduction steps involved in the application of the phase gate P upon the vector **true**, as a motivating example for these rules:

$$\begin{aligned} & \left((\text{false} \triangleright \text{false}) + \text{true} \triangleright \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right). \text{true} \right) * \text{true} \\ \longrightarrow^* & (\text{false} \triangleright \text{false}) * \text{true} + (\text{true} \triangleright \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right). \text{true}) * \text{true} \\ \longrightarrow^* & (\text{false} \bullet \text{true}).\text{false} + (\text{true} \bullet \text{true}). \left(\left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right). \text{true} \right) \\ \longrightarrow^* & 0.\text{false} + \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right). \text{true} \\ \longrightarrow^* & \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right). \text{true}. \end{aligned}$$

All of the three gates forming a universal set for quantum computation are trivially expressed as terms in this notation:

$$\begin{aligned} CNOT &= (\text{false} \otimes \text{false}) \triangleright (\text{false} \otimes \text{false}) \\ &\quad + (\text{false} \otimes \text{true}) \triangleright (\text{false} \otimes \text{true}) \\ &\quad + (\text{true} \otimes \text{false}) \triangleright (\text{true} \otimes \text{true}) \\ &\quad + (\text{true} \otimes \text{true}) \triangleright (\text{true} \otimes \text{false}) \\ H &= \left(\text{false} \triangleright \frac{1}{\sqrt{2}}.(\text{false} + \text{true}) \right) + \left(\text{true} \triangleright \frac{1}{\sqrt{2}}.(\text{false} - \text{true}) \right) \\ P &= (\text{false} \triangleright \text{false}) + \left(\text{true} \triangleright \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right). \text{true} \right). \end{aligned}$$

4.2 Rules

Since \triangleright is just another type of tensor product, bilinearity applies (see figure 5. Notice the conjugation of the λ scalar, denoted $\bar{\lambda}$, easily implemented in the TRS). Other than its left-hand-side antilinearity, the particularity of \triangleright is the reduction it

Figure 5: MATCHING OPERATORS BILINEARITY

$$\begin{aligned}
 (\mathbf{t} + \mathbf{u}) \triangleright \mathbf{v} &\longrightarrow \mathbf{t} \triangleright \mathbf{v} + \mathbf{u} \triangleright \mathbf{v} \\
 \mathbf{t} \triangleright (\mathbf{v} + \mathbf{w}) &\longrightarrow \mathbf{t} \triangleright \mathbf{v} + \mathbf{t} \triangleright \mathbf{w} \\
 (\lambda.\mathbf{u}) \triangleright \mathbf{v} &\longrightarrow \bar{\lambda}.(\mathbf{u} \triangleright \mathbf{v}) \\
 \mathbf{u} \triangleright (\mu.\mathbf{v}) &\longrightarrow \mu.(\mathbf{u} \triangleright \mathbf{v}) \\
 \mathbf{0} \triangleright \mathbf{u} &\longrightarrow \mathbf{0} \\
 \mathbf{u} \triangleright \mathbf{0} &\longrightarrow \mathbf{0} \\
 (\mathbf{u} + \mathbf{v}) * \mathbf{w} &\longrightarrow \mathbf{u} * \mathbf{w} + \mathbf{v} * \mathbf{w} \\
 (\lambda.\mathbf{u}) * \mathbf{v} &\longrightarrow \lambda.(\mathbf{u} * \mathbf{v}) \\
 \mathbf{u} * (\mathbf{v} + \mathbf{w}) &\longrightarrow \mathbf{u} * \mathbf{v} + \mathbf{u} * \mathbf{w} \\
 \mathbf{u} * (\lambda.\mathbf{v}) &\longrightarrow \lambda.(\mathbf{u} * \mathbf{v}) \\
 \mathbf{0} * \mathbf{u} &\longrightarrow \mathbf{0} \\
 \mathbf{u} * \mathbf{0} &\longrightarrow \mathbf{0} \\
 (\mathbf{t} + \mathbf{u}) \bullet \mathbf{v} &\longrightarrow \mathbf{t} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{v} \\
 \mathbf{t} \bullet (\mathbf{v} + \mathbf{w}) &\longrightarrow \mathbf{t} \bullet \mathbf{v} + \mathbf{t} \bullet \mathbf{w} \\
 (\lambda.\mathbf{u}) \bullet \mathbf{v} &\longrightarrow \bar{\lambda}.(\mathbf{u} \bullet \mathbf{v}) \\
 \mathbf{u} \bullet (\mu.\mathbf{v}) &\longrightarrow \mu.(\mathbf{u} \bullet \mathbf{v}) \\
 \mathbf{0} \bullet \mathbf{u} &\longrightarrow \mathbf{0} \\
 \mathbf{u} \bullet \mathbf{0} &\longrightarrow \mathbf{0}
 \end{aligned}$$

Figure 6: MATCHING OPERATOR AND THE SCALAR PRODUCT

$$\begin{aligned}
 (\mathbf{t} \triangleright \mathbf{u}) * \mathbf{v} &\longrightarrow (\mathbf{t} \bullet \mathbf{v}).\mathbf{u} \\
 (\mathbf{t} \otimes \mathbf{u}) \bullet (\mathbf{v} \otimes \mathbf{w}) &\longrightarrow (\mathbf{t} \bullet \mathbf{v}) \times (\mathbf{u} \bullet \mathbf{w}) \\
 (\mathbf{t} \triangleright \mathbf{u}) \bullet (\mathbf{v} \triangleright \mathbf{w}) &\longrightarrow \overline{(\mathbf{t} \bullet \mathbf{v})} \times (\mathbf{u} \bullet \mathbf{w}) \\
 \mathbf{true} \bullet \mathbf{true} &\longrightarrow 1 \\
 \mathbf{true} \bullet \mathbf{false} &\longrightarrow 0 \\
 \mathbf{false} \bullet \mathbf{true} &\longrightarrow 0 \\
 \mathbf{false} \bullet \mathbf{false} &\longrightarrow 1
 \end{aligned}$$

induces when placed left of an application symbol $*$, as described in figure 6. For definiteness, we may also add the rules in figure 7.

Notice that for now programming language does not use any variables. All functions are defined by adding elementary functions mapping base vectors to base vectors. And all functions are linear by constructions. Moreover until Section 7 we do not worry about normalization and unitarity conditions. For now the \triangleright notation provides exactly what is needed: linear operations can be encoded as sums of tensor states describing which vector is associated to which [3][8].

5 Lambda calculus construct

We now turn to the last ingredient of our language: the lambda-calculus constructs. In this case we will need to introduce bound variables. These bound variables will be handled through the use of de Bruijn indices. We shall also be more careful, as lambda-calculus usually allows to express non linear functions, whilst we have to restrict ourselves to linear ones.

5.1 Cloning

As we seek to provide a language of linear operator, we must pay attention to the fact that duplication of a vector can lead to nonlinear evolutions. For instance, whatever formalism we choose for quantum theory (vectors or density matrices), quantum operations act linearly upon their input states. This, in turn, implies that quantum states cannot be *cloned*. Indeed such an evolution acts upon a qubit as

Figure 7: ORTHOGONALITY RULES

$$\begin{aligned}
(\mathbf{t} \otimes \mathbf{u}) \bullet (\mathbf{v} \triangleright \mathbf{w}) &\longrightarrow 0 \\
(\mathbf{t} \otimes \mathbf{u}) \bullet \mathbf{true} &\longrightarrow 0 \\
(\mathbf{t} \otimes \mathbf{u}) \bullet \mathbf{false} &\longrightarrow 0 \\
(\mathbf{t} \triangleright \mathbf{u}) \bullet (\mathbf{v} \otimes \mathbf{w}) &\longrightarrow 0 \\
(\mathbf{t} \triangleright \mathbf{u}) \bullet \mathbf{true} &\longrightarrow 0 \\
(\mathbf{t} \triangleright \mathbf{u}) \bullet \mathbf{false} &\longrightarrow 0 \\
\mathbf{true} \bullet (\mathbf{v} \otimes \mathbf{w}) &\longrightarrow 0 \\
\mathbf{true} \bullet (\mathbf{v} \triangleright \mathbf{w}) &\longrightarrow 0 \\
\mathbf{false} \bullet (\mathbf{v} \otimes \mathbf{w}) &\longrightarrow 0 \\
\mathbf{false} \bullet (\mathbf{v} \triangleright \mathbf{w}) &\longrightarrow 0
\end{aligned}$$

follows:

$$\begin{aligned}
(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle &\xrightarrow{\text{CLONE}} (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\
\begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix} &\xrightarrow{\text{CLONE}} \begin{pmatrix} \alpha^2 \\ \alpha\beta \\ \alpha\beta \\ \beta^2 \end{pmatrix},
\end{aligned}$$

which cannot be linear (a more formal discussion can be found in [26]). Cloning should be distinguished from *copying* however, as we now illustrate once more on qubit:

$$\begin{aligned}
|0\rangle \otimes |0\rangle &\xrightarrow{\text{COPY}} |0\rangle \otimes |0\rangle \\
|1\rangle \otimes |0\rangle &\xrightarrow{\text{COPY}} |1\rangle \otimes |1\rangle \\
(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle &\xrightarrow{\text{COPY}} \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle
\end{aligned}
\quad \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix} \xrightarrow{\text{COPY}} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix}.$$

Such an evolution is perfectly valid, and in the above case it may be implemented as a single application of the quantum gate *CNOT*, which is of course both linear and unitary.

In classical functional languages terms such as $\lambda x.f(x, x)$, with $(\lambda x.f(x, x) \ t) \xrightarrow{\beta} f(t, t)$, are crucial for the expressiveness. Recursion, for instance, relies upon such terms, and is absolutely necessary for universality. When designing a quantum functional language we therefore face a choice:

- Either we prevent terms such as $\lambda x.f(x, x)$ from being applied to quantum states — thereby ensuring that no quantum cloning is allowed. But we must authorize their applications upon “classical terms” for expressiveness. As a consequence the language must be able to keep track of quantum resources versus classical resources. This is the approach followed by Van Tonder [23].
- Or we allow terms such as $\lambda x.f(x, x)$ from being applied to quantum states — only to be interpreted as a quantum copy. We may still want to keep track of quantum resources versus classical resources, but not for the purpose of forbidding cloning. This is the approach we take.

For now the latter option seems preferable, since it models the *dos* and *don'ts* of the linearity requirement more closely, whilst keeping the calculus to a minimum. Moreover copying, as we now show, can be imposed over cloning by the semantics of the calculus alone. Thus, the fact that classical states can be cloned is proved and not postulated in our language.

5.2 Substitution of de Bruijn indices

The point of the previous discussion is that we can only duplicate basis vectors. Informally

$$\begin{aligned}
& (\lambda x.(x \otimes x)) * \mathbf{true} \longrightarrow^* \mathbf{true} \otimes \mathbf{true} \quad \text{is OK;} \\
& (\lambda x.(x \otimes x)) * (\mathbf{false} + \mathbf{true}) \longrightarrow^* ((\lambda x.(x \otimes x)) * \mathbf{false}) + ((\lambda x.(x \otimes x)) * \mathbf{true}) \\
& \quad \longrightarrow^* (\mathbf{true} \otimes \mathbf{true}) + (\mathbf{false} \otimes \mathbf{false}) \quad \text{is OK;} \\
& (\lambda x.(x \otimes x)) * (\mathbf{false} + \mathbf{true}) \longrightarrow^* (\mathbf{false} + \mathbf{true}) \otimes (\mathbf{false} + \mathbf{true}) \quad \text{is not OK.}
\end{aligned}$$

Again another way to grasp this idea is to realize that faced with a term of the form $(\lambda x.t) * (u + v)$, one could either start by proceeding to the substitution, or start by applying the right-hand-side linearity of $*$, leading to two different results. So that operations remain linear, we must favour the right-hand-side linearity of $*$ over substitution. The rules of figure 8 accomplish exactly that. The three first rules are the most straightforward, they invoke the linearity of the vector to be substituted. The three following rules treat the base cases, when the vector to be substituted is down to a basic state. The last two rules handle the more subtle case of tensor states $\mathbf{u} \otimes \mathbf{v}$ or $\mathbf{u} \triangleright \mathbf{v}$. In a word the trick is to treat

$$(\lambda x.(\dots x \dots)) * (\mathbf{u} \otimes \mathbf{v}) \quad \text{as} \quad \left(\lambda x.(\lambda y.(\dots x \otimes y \dots)) * \mathbf{u} \right) * \mathbf{v}$$

Figure 8: ENFORCING LINEARITY OVER SUBSTITUTION

$$\begin{aligned}
L(\mathbf{u}) * \mathbf{v} &\longrightarrow (\mathbf{u} \text{ of } \mathbf{v}) \\
\mathbf{t} \text{ of } (r.\mathbf{v}) &\longrightarrow r.(\mathbf{t} \text{ of } \mathbf{v}) \\
\mathbf{t} \text{ of } (\mathbf{v} + \mathbf{w}) &\longrightarrow (\mathbf{t} \text{ of } \mathbf{v}) + (\mathbf{t} \text{ of } \mathbf{w}) \\
\mathbf{t} \text{ of } \mathbf{true} &\longrightarrow \mathbf{t} \text{ bof subst}(\mathbf{true}) \\
\mathbf{t} \text{ of } \mathbf{false} &\longrightarrow \mathbf{t} \text{ bof subst}(\mathbf{false}) \\
\mathbf{t} \text{ of } \mathbf{0} &\longrightarrow \mathbf{t} \text{ bof subst}(\mathbf{0}) \\
\mathbf{t} \text{ of } (\mathbf{v} \otimes \mathbf{w}) &\longrightarrow \\
&\left(\left(((\mathbf{t} \text{ bof } \uparrow(\uparrow)) \text{ bof } \uparrow(\uparrow)) \text{ bof subst}(\mathbf{var}(0) \otimes \mathbf{var}(S(0))) \right) \text{ of } \mathbf{v} \right) \text{ of } \mathbf{w} \\
\mathbf{t} \text{ of } (\mathbf{v} \triangleright \mathbf{w}) &\longrightarrow \\
&\left(\left(((\mathbf{t} \text{ bof } \uparrow(\uparrow)) \text{ bof } \uparrow(\uparrow)) \text{ bof subst}(\mathbf{var}(0) \triangleright \mathbf{var}(S(0))) \right) \text{ of } \mathbf{v} \right) \text{ of } \mathbf{w}
\end{aligned}$$

with y a fresh variable, and then proceed recursively.

Once the vector to be substituted is a basic state, we can safely proceed to the substitution using a calculus of explicit substitutions [1, 10, 16]. Here we have chosen to represent variables by their de Bruijn indices, i.e. each variable is now an integer number corresponding to number of binders (“ L ” or “ λ ” symbols) one must go through before reaching the binding occurrence. For instance

$$\lambda x.(\lambda y.(x \otimes y)) \text{ is encoded as } L(L(\mathbf{var}(1) \otimes \mathbf{var}(0)))$$

since there is one λ symbol lying between x and the binding occurrence of x . This variable numbering scheme is often used for implementing functional languages. Notice how in this scheme a variable may be denoted differently depending upon its position in the term (i.e. depending upon how far it lies from its binding occurrence). The rules of figure 9 implement this mechanism.

Thus there are two ways to use the λ -calculus to define a linear map in a vectorial space of countable dimension over a countable field. The first is to interpret the λ -terms as functions mapping vectors to vectors and in this case we need extra constraints to enforce linearity. The second is to interpret the λ -terms as functions mapping base vectors to base vectors and to extend it to the full space with extra computation rules. This solution is advantageous as it requires no restriction on the λ -terms.

Notice that our language allows a restricted form of higher-order programming

Figure 9: EXPLICIT SUBSTITUTION OF DE BRUIJN INDICES

$$\begin{aligned}
(\mathbf{t} + \mathbf{u}) \text{ bof } \mathbf{s} &\longrightarrow (\mathbf{t} \text{ bof } \mathbf{s}) + (\mathbf{u} \text{ bof } \mathbf{s}) \\
(r.\mathbf{u}) \text{ bof } \mathbf{s} &\longrightarrow (r \text{ bof } \mathbf{s}).(\mathbf{u} \text{ bof } \mathbf{s}) \\
(\mathbf{t} \bullet \mathbf{u}) \text{ bof } \mathbf{s} &\longrightarrow (\mathbf{t} \text{ bof } \mathbf{s}) \bullet (\mathbf{u} \text{ bof } \mathbf{s}) \\
(r + s) \text{ bof } \mathbf{s} &\longrightarrow (r \text{ bof } \mathbf{s}) + (s \text{ bof } \mathbf{s}) \\
(r \times s) \text{ bof } \mathbf{s} &\longrightarrow (r \text{ bof } \mathbf{s}) \times (s \text{ bof } \mathbf{s}) \\
r \text{ bof } \mathbf{s} &\longrightarrow r \\
(\mathbf{t} \otimes \mathbf{u}) \text{ bof } \mathbf{s} &\longrightarrow (\mathbf{t} \text{ bof } \mathbf{s}) \otimes (\mathbf{u} \text{ bof } \mathbf{s}) \\
(\mathbf{t} \triangleright \mathbf{u}) \text{ bof } \mathbf{s} &\longrightarrow (\mathbf{t} \text{ bof } \mathbf{s}) \triangleright (\mathbf{u} \text{ bof } \mathbf{s}) \\
(\mathbf{t} * \mathbf{u}) \text{ bof } \mathbf{s} &\longrightarrow (\mathbf{t} \text{ bof } \mathbf{s}) * (\mathbf{u} \text{ bof } \mathbf{s}) \\
L(\mathbf{t}) \text{ bof } \mathbf{s} &\longrightarrow L(\mathbf{t} \text{ bof } \uparrow(\mathbf{s})) \\
\mathbf{0} \text{ bof } \mathbf{s} &\longrightarrow \mathbf{0} \\
\mathbf{false} \text{ bof } \mathbf{s} &\longrightarrow \mathbf{false} \\
\mathbf{true} \text{ bof } \mathbf{s} &\longrightarrow \mathbf{true} \\
\mathbf{var}(0) \text{ bof subst}(\mathbf{v}) &\longrightarrow \mathbf{v} \\
\mathbf{var}(S(p)) \text{ bof subst}(\mathbf{v}) &\longrightarrow \mathbf{var}(p) \\
\mathbf{var}(0) \text{ bof } \uparrow(\mathbf{s}) &\longrightarrow \mathbf{var}(0) \\
\mathbf{var}(S(p)) \text{ bof } \uparrow(\mathbf{s}) &\longrightarrow (\mathbf{var}(p) \text{ bof } \mathbf{s}) \text{ bof } \uparrow \\
\mathbf{var}(p) \text{ bof } \uparrow &\longrightarrow \mathbf{var}(S(p))
\end{aligned}$$

where a function F defined by a λ -term can take another function g as its argument, provided the function g is expressed with the matching construct, but not if it is expressed as a λ -term also. The extension of this language to full higher-order programming is left for future work.

6 An example

The Deutch-Jozsa algorithm can be defined in our language as follows

$$DJ = \lambda x ((Cross * H * H) * (x * ((Cross * H * H) * (\mathbf{false} \otimes \mathbf{true}))))$$

Figure 10: Summary

Computational scalars	(rules Fig. 3, 4 and 1)
0	null scalar
1	unit scalar
+	scalar sum
×	scalar product
Vectorial space	(rules Fig. 1)
0	null vector
false, true	base vectors
+	vector sum
.	product of a vector by a scalar
Tensorial product	(rules Fig. 2)
⊗	tensorial product
Matching operators	(rules Fig. 5, 6)
▷	matching construct
*	function application
•	scalar product
Lambda calculus	(rules Fig. 8, 9)
<i>L</i>	lambda abstraction
var	variables
of, bof, subst, ↑, ↑↑	explicit substitutions constructs

where

$$\begin{aligned}
 Cross = & \lambda x \lambda y (\\
 & (\mathbf{false} \otimes \mathbf{false}) \triangleright ((x * \mathbf{false}) \otimes (y * \mathbf{false})) \\
 & + (\mathbf{false} \otimes \mathbf{true}) \triangleright ((x * \mathbf{false}) \otimes (y * \mathbf{true})) \\
 & + (\mathbf{true} \otimes \mathbf{false}) \triangleright ((x * \mathbf{true}) \otimes (y * \mathbf{false})) \\
 & + (\mathbf{true} \otimes \mathbf{true}) \triangleright ((x * \mathbf{true}) \otimes (y * \mathbf{true})))
 \end{aligned}$$

i.e. using de Bruijn indices

$$DJ = L((Cross * H * H) * (\mathbf{var}(0) * ((Cross * H * H) * (\mathbf{false} \otimes \mathbf{true}))))$$

where

$$\begin{aligned}
Cross = L(L(\\
& (\mathbf{false} \otimes \mathbf{false}) \triangleright ((\mathbf{var}(S(0)) * \mathbf{false}) \otimes (\mathbf{var}(0) * \mathbf{false})) \\
& + (\mathbf{false} \otimes \mathbf{true}) \triangleright ((\mathbf{var}(S(0)) * \mathbf{false}) \otimes (\mathbf{var}(0) * \mathbf{true})) \\
& + (\mathbf{true} \otimes \mathbf{false}) \triangleright ((\mathbf{var}(S(0)) * \mathbf{true}) \otimes (\mathbf{var}(0) * \mathbf{false})) \\
& + (\mathbf{true} \otimes \mathbf{true}) \triangleright ((\mathbf{var}(S(0)) * \mathbf{true}) \otimes (\mathbf{var}(0) * \mathbf{true}))))
\end{aligned}$$

Then it can be checked that the term $DJ * CNOT$ reduces to $\mathbf{true} \otimes \mathbf{true}$ whose first component is indeed the exclusive disjunction of $\mathbf{not}(\mathbf{true})$ and $\mathbf{not}(\mathbf{false})$.

7 Discussion

7.1 Unitarity

In its simplest formulation quantum theory only allows unitary evolutions, i.e. vectors evolve in time according to square matrices U verifying $U^\dagger U = \mathbb{I}$. In this framework it is impossible to delete, say, a qubit:

$$\begin{aligned}
|0\rangle & \xrightarrow{\text{ERASE}} |0\rangle \\
|1\rangle & \xrightarrow{\text{ERASE}} |0\rangle \\
(\alpha|0\rangle + \beta|1\rangle) & \xrightarrow{\text{ERASE}} |0\rangle,
\end{aligned}$$

the evolution is not injective and therefore not unitary. Von Neumann's projective measurements help us only partially: if the vector is measured in the canonical basis, and when this measurement yields outcome "0", then the qubit undergoes the above exact dynamics. But this will only occur with probability $|\alpha|^2$.

The ERASE operation is however perfectly physical, as one can always ignore a qubit and focus upon another, taken to be in state $|0\rangle$. Moreover the process needs not be probabilistic. There are two well-established formulations of quantum theory which cater for this possibility:

- The *generalized measurement* formalism unifies quantum evolutions and quantum measurements as one single object. Mathematically a generalized measurement is given by a set of matrices $\{M_m\}$ verifying

$$\sum_m M_m^\dagger M_m \ll Id. \tag{2}$$

A vector v will then evolve in time according to the matrix M_m with probability $p_m = |M_m v|^2$ — in which case we shall say that outcome “ m ” has occurred. As an example the following generalized measurement performs the ERASE operation:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

For a more detailed presentation of these concepts the reader is referred to [18], page 84.

- The *density matrix* formalism represents quantum states as positive matrices instead of vectors. These evolve in time according to Completely Positive-preserving maps, i.e. operations of the form

$$\rho \longmapsto \sum_m M_m \rho M_m^\dagger \quad \text{with probability} \quad p_m = \text{Tr}(M_m \rho M_m^\dagger)$$

and verifying Equation (2). This framework is traditional when dealing with open quantum systems, and therefore well appropriate as one discards a qubit.

In classical functional languages terms such as $\lambda y.(\lambda x.x)$, with $(\lambda y.(\lambda x.x)\mathbf{t}) \xrightarrow{\beta} \lambda x.x$, are commonly used. Boolean values and branching, for instance, are encoded in such manners. Although convenient, non-injective functions are not absolutely necessary, for reversible computation can be both universal and efficient [5]. Whether a quantum functional language should allow erasure or not must therefore depend upon which of the three above mentioned formulation of quantum theory gets chosen.

If we adopt the simplest formulation of quantum theory, our language must be restricted to operation which are not only linear but also unitary. At first sight this seems feasible by imposing the standard $U^\dagger U = \mathbb{I}$ condition upon the matching constructs of section 4, and the *relevance* condition upon the λ -terms of 5 (i.e. for all term $\lambda x.t$ the variable x must occur at least once in t). This remains a subject for future work.

7.2 Linearity?

Linear logic appears in [13] as a mean to express and prove properties of dynamical systems where the *consumption* of resources is important. The standard example (price updated) is $A \equiv \text{“I have 6€”}$, $B \equiv \text{“I have a paquet of Gauloises”}$,

and the statement $A \multimap B$ to express the possibility of using up A to obtain B . With \otimes now expressing a conjunction, it is clear one cannot have $A \multimap (B \otimes B)$, since this would mean buying two paquets for the price of one. Neither can we have $A \otimes A \multimap B$: we must get something for our money — at worse the feeling of getting cheated. Therefore the rules governing symbols \multimap , \otimes differ from those of classical logic for \Rightarrow , \wedge . Unless there is an abundance of resources (denoted by the exclamation mark “!”), in which case they coincide again: $!A \multimap (B \otimes B \dots)$. Whilst considering this point the father of Linear logic has the following thought [14]: “*Classical logic appears to be the logic of macro-actions, as opposed to linear logic which would be a logic of micro-actions. The unusual character of linear logic may therefore be considered similar to the strange character of micro-mechanics, i.e. quantum mechanics.*”.

Specifications expressed in linear logics can be seen as types for programs expressed in linear λ -calculus. In the linear λ -calculus one distinguishes *linear resources*, which may not be copied nor discarded, from *nonlinear resources*, which are denoted by the exclamation mark “!” and whose fate is not subjected to particular restrictions. Van Tonder’s quantum λ -calculus (λ_q) is founded upon these ideas. As we have mentioned in 5.1, he uses this well-established framework in order to distinguish quantum resources (treated as linear) from classical resources (treated as nonlinear):

$$\begin{aligned}
(\lambda_q) \quad & t ::= x \mid \lambda x.t \mid (t \ t) \mid c \mid !t \mid \lambda!x.t \\
& c ::= 0 \mid 1 \mid H \mid CNOT \mid P \\
& \text{(together with the well-formedness rules of the classical linear} \\
& \text{ } \lambda \text{ calculus)} \\
(\mathcal{R}_q) \quad & (\lambda x.t \ s) \xrightarrow{\beta} t[s/x] \\
& H \ 0 \longrightarrow 0 + 1 \\
& H \ 1 \longrightarrow 0 - 1 \\
& \vdots
\end{aligned}$$

Here the well-formedness conditions of the classical linear calculus (which prevents linear terms from being discarded), together with !-suspension (which stops quantum terms from being treated as nonlinear) maintain unitarity throughout the reductions.

The connection between the linear λ -calculus and quantum functional languages is striking. It comes at a price however: the λ_q -calculus remains hetero-

geneous, i.e. a juxtaposition of quantum resources (linear resources) and classical resources (nonlinear resources). In some sense it is twice linear, both in the sense of linear λ -calculus and linear algebra, and thus in some sense overrestricted. In particular it forbids both the cloning of quantum data (which needs be done) and the copying of quantum data (which needs not be done). As a consequence its control flow remains inherently based upon classical resources.

The *linear-algebraic λ -calculus* constructed in this paper is homogeneous, i.e. it does not draw a line between quantum resources and classical resources (the latter are merely thought of as basis states of the former). Moreover it exhibits only one notion of linearity, which is that of linear algebra. Thus cloning remains disallowed (just by the semantics) but not copy. Control flow is still provided as a consequence. These results seem to open the way to a linear algebraic interpretation of linear logic, in the spirit of Girard's Geometry of interaction, although much work remains ahead in order to strengthen this connection.

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